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Abstract: The concept of (A, B)-invariant subspace (or controlled invariant) of a linear dynamical system is extended to linear systems over the max-plus semiring. Although this extension presents several difficulties, which are similar to those encountered in the same kind of extension to linear dynamical systems over rings, it appears capable of providing solutions to many control problems like in the cases of linear systems over fields or rings. Sufficient conditions are given for computing the maximal (A, B)-invariant subspace contained in a given space and the existence of linear state feedbacks is discussed. An application to the study of transportation networks which evolve according to a timetable is considered.

Key-words: invariant spaces, geometric control, max-plus algebra, Discrete Event Systems

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Espaces (A, B)-invariants max-plus et commande de systèmes à événements discrets temporisés

Résumé : Nous étendons la notion de sous-espace (A, B)-invariant au cas des systèmes linéaires sur le semi-anneau max-plus. Bien que cette extension présente des difficultés analogues au cas des systèmes linéaires à coefficients dans des anneaux, elle permet d'aborder des problèmes de contrôle de systèmes à événements discrets. Nous donnons des conditions suffisantes, sous lesquelles il est possible de calculer le sous-espace (A, B)-invariant maximal contenu dans un espace donné. Nous montrons aussi comment déterminer un feedback linéaire associé, lorsqu'un tel feedback existe. Nous appliquons cette approche à la synthèse d'horaire pour un exemple de réseau de transport.

Mots-clés : Espaces invariants, approche géométrique, algèbre max-plus, systèmes à événements discrets.

1 Introduction

The geometric approach to the theory of linear dynamical systems has provided deep insights and elegant solutions to many control problems, such as the disturbance decoupling problem, the block decoupling problem, and the model matching problem (see [Won85] and the references therein). The concept of (A,B)-invariant subspace (or controlled invariant subspace, see [BM91]) has played a significant role in the development of this approach.

It is natural to try to apply the same kind of methods to discrete event systems. Several mathematical models have been proposed, see in particular [CLO95] for a survey of the following approaches. Ramadge and Wonham [RW87] initiated the logical, language-theoretic approach, in which the precise ordering of the events is of interest and time does not play an explicit role. This theory addresses the synthesis of controllers in order to satisfy some qualitative specifications on the admissible orderings of the events. The max-plus algebra based control approach initiated by Cohen et al. [CDQV85], in which, in addition to the ordering, the timing of the events plays an essential role. A third approach is the perturbation analysis of Cassandras and Ho [CH83], which deals with stochastic timed discrete event systems.

The max-plus semiring is the set $\mathbb{R}\cup\{-\infty\}$, equipped with max as addition and the usual sum as multiplication. Linear dynamical systems with coefficients in the max-plus semiring turn out to be useful for modeling and analyzing many discrete event dynamic systems subject to synchronization constraints (see [BCOQ92]). Among these, we can mention some manufacturing systems (Cohen et al. [CDQV85]), computer networks (Le Boudec and Thiran [BT01]) and transportation networks (Olsder et al. [OSG98], Braker [Bra91, Bra93], and de Vries et al. [dDD98]). Many results from linear system theory have been extended to systems with coefficients in the max-plus semiring, such as the connection between spectral theory and stability questions (see [CMQV89]) or transfer series methods (see [BCOQ92]). Several interesting control problems have also been studied by, for example, Boimond et al. [BCFH99, BFHM00], Cottenceau et al. [CHMSM03] and Lhommeau [Lho03]. In contrast to the approach presented here, which is based on state space representation, their approach uses transfer series and residuation methods and therefore deals with different types of specifications.

This motivates the attempt to extend the geometric approach, and in particular the concept of (A,B)-invariant subspace, to the theory of linear dynamical systems over the max-plus semiring, a question which is raised in [CGQ99]. The same kind of generalization, which was initiated by Hautus, Conte and Perdon, has been widely studied for linear dynamical systems over rings (see [Hau82, Hau84, CP94, CP95, Ass99, ALP99]). In this paper we will see that the extension of the geometric approach to linear systems over the max-plus semiring presents similar difficulties to those encountered in dealing with coefficients in a ring rather than coefficients in a field.

To illustrate one of the possible applications of the results presented in this paper, we apply the methods presented here to the study of transportation networks which evolve according to a timetable. Max-plus linear models for transportation networks have been studied by several authors, see for example [OSG98, Bra91, Bra93, dDD98]. Let us consider

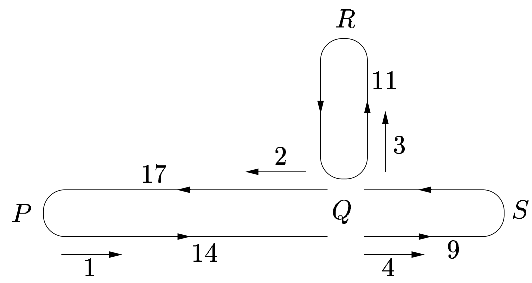

Figure 1: A simple transportation network

the simple railway network given in Figure 1, which has been borrowed from [dDD98]. In this network there is a train service from station P to station S via station Q and vice versa, and there is also a service from station Q to station R and back. The traveling times for each of the four possible destinations are indicated in the figure. We will assume that the following conditions are satisfied. A first condition is that at station Q the trains coming from stations P and S have to ensure a connection to the train which leaves for destination R and vice versa. The second condition is that a train cannot leave before its scheduled departure time which is given by a timetable. If we assume that a train leaves as soon as all the previous conditions have been satisfied, then the evolution of the transportation network can be described by a max-plus linear dynamical system where the scheduled departure times can be seen as controls (see Section 6). We will see that the tools presented in this paper can be used to analyze this kind of network. For example, it is possible to determine whether there exists a timetable that satisfies such conditions as the following. A first condition could be that the time between two consecutive departures of trains in the same direction be less than a certain given bound. As a second condition we could require that the time that people have to wait to make some connections be less than another given bound. Of course, more general specifications could be analyzed. We show how to compute a timetable which satisfies these requirements when they exist. For instance, suppose that in the railway network given in Figure 1 we want the time between two consecutive departures of trains in the same direction to be less than 15 time units and the maximal time that people have to wait to make any connection to be less than 4 time units. In Section 6 we show that this is possible and give a timetable which satisfies these requirements.

This paper is organized as follows. In Section 2, after a short introduction to max-plus type semirings, we introduce the concept of geometrically (A, B)-invariant semimodules and generalize to max-plus algebra Wonham fixed point algorithm (see [Won85]) which is used to compute the maximal (A, B)-invariant subspace contained in a given space. In Section 3 we introduce the concept of volume of a semimodule and study its properties. In Section 4 we use volume arguments to show that the fixed point algorithm introduced in Section 2 converges in a finite number of steps for an important class of semimodules. In Section 5 we consider the concept of algebraically (A, B)-invariant semimodules and give a method to decide whether a finitely generated semimodule is algebraically (A, B)-invariant. Finally, in Section 6 we apply the methods given in this paper to the study of transportation networks which evolve according to a timetable.

Let us finally mention that some of the results presented here were announced in [GK03] and considered in [Kat03].

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2 Geometrically (A, B)-invariant semimodules

Let us first recall some definitions and results. A *monoid* is a set equipped with an associative internal composition law which has a (two sided) neutral element. A *semiring* is a set $\mathcal{S}$ equipped with two internal composition laws $\oplus$ and $\otimes$, called addition and multiplication respectively, such that $\mathcal{S}$ is a commutative monoid for addition, $\mathcal{S}$ is a monoid for multiplication, multiplication distributes over addition, and the neutral element for addition is absorbing for multiplication. We will sometimes denote by $(\mathcal{S}, \oplus, \otimes, \varepsilon, e)$ the semiring $\mathcal{S}$, where ε and e represent the neutral elements for addition and for multiplication respectively. We say that a semiring $\mathcal{S}$ is *idempotent* if $x \oplus x = x$ for all $x \in \mathcal{S}$. In this paper, we are mostly interested in some variants of the max-plus semiring $\mathbb{R}_{\max}$, which is the set $\mathbb{R} \cup \{-\infty\}$ equipped with $\oplus = \max$ and $\otimes = +$ (see [Pin98] for an overview). Some of these variants can be obtained by noting that to any submonoid $(M, +)$ of $(\mathbb{R}, +)$ is associated a semiring $M_{\max}$ whose set of elements is $M \cup \{-\infty\}$ and laws $\oplus = \max$ and $\otimes = +$. Symmetrically, we can consider the semiring $M_{\min}$ with the set of elements $M \cup \{+\infty\}$ and laws $\oplus = \min$ and $\otimes = +$. For instance, taking $M = \mathbb{Z}$ we get the semiring $\mathbb{Z}_{\max} = (\mathbb{Z} \cup \{-\infty\}, \max, +)$, which is the main semiring we are going to work with, and taking $M = \mathbb{N}$ we get the semiring $\mathbb{N}_{\min} = (\mathbb{N} \cup \{+\infty\}, \min, +)$, which is known as the *tropical semiring* after the work of Simon (see [Sim78]). Recall that an idempotent semiring $(\mathcal{S}, \oplus, \otimes)$ is equipped with the *natural order*: $x \preceq y \iff x \oplus y = y$ (see for example [BCOQ92]). We can also add a maximal element for the natural order to the semirings $M_{\max}$ and $M_{\min}$, obtaining in this way the semirings $\overline{M}_{\max} = (M \cup \{\pm\infty\}, \max, +)$ and $\overline{M}_{\min} = (M \cup \{\pm\infty\}, \min, +)$, respectively. Note that, in the semirings $\overline{M}_{\max}$ and $\overline{M}_{\min}$, the value of $(-\infty) + (+\infty) = (+\infty) + (-\infty)$ is determined by the fact that the neutral element for addition is absorbing for multiplication. Then, we know that $(-\infty) + (+\infty) = (+\infty) + (-\infty) = -\infty$ in $\overline{M}_{\max}$ and $(-\infty) + (+\infty) = (+\infty) + (-\infty) = +\infty$ in $\overline{M}_{\min}$.

A (left) *semimodule* over a semiring $(\mathcal{S}, \oplus, \otimes, \varepsilon_{\mathcal{S}}, e)$ is a commutative monoid $(\mathcal{X}, \hat{\oplus})$, with neutral element $\varepsilon_{\mathcal{X}}$, equipped with a map $\mathcal{S} \times \mathcal{X} \to \mathcal{X}$, $(\lambda, x) \to \lambda \cdot x$ (left action), which satisfies:

$$(\lambda \otimes \mu) \cdot x = \lambda \cdot (\mu \cdot x) \; ,$$
$$\lambda \cdot (x \hat{\oplus} y) = \lambda \cdot x \hat{\oplus} \lambda \cdot y \; ,$$

$$
\begin{aligned}
(\lambda \oplus \mu) \cdot x &= \lambda \cdot x \,\hat{\oplus}\, \mu \cdot x \ , \\
\varepsilon_{\mathcal{S}} \cdot x &= \varepsilon_{\mathcal{X}} \ , \\
\lambda \cdot \varepsilon_{\mathcal{X}} &= \varepsilon_{\mathcal{X}} \ , \\
e \cdot x &= x \ ,
\end{aligned}
$$

for all $x, y \in \mathcal{X}$ and $\lambda, \mu \in \mathcal{S}$. We will usually use concatenation to denote both the multiplication of $\mathcal{S}$ and the left action, and we will denote by ε both the zero element $\varepsilon_{\mathcal{S}}$ of $\mathcal{S}$ and the zero element $\varepsilon_{\mathcal{X}}$ of $\mathcal{X}$. A *subsemimodule* of $\mathcal{X}$ is a subset $\mathcal{Z} \subset \mathcal{X}$ such that $\lambda x \hat{\oplus} \mu y \in \mathcal{Z}$, for all $x, y \in \mathcal{Z}$ and $\lambda, \mu \in \mathcal{S}$. In this paper, we will mostly consider subsemimodules of the *free semimodule* $\mathcal{S}^n$, which is the set of n-dimensional vectors over $\mathcal{S}$, equipped with the internal law $(x \hat{\oplus} y)_i = x_i \oplus y_i$ and the left action $(\lambda x)_i = \lambda \otimes x_i$. If $G \subset \mathcal{X}$, we will denote by $\operatorname{span} G$ the subsemimodule of $\mathcal{X}$ generated by G, that is, the set of all $x \in \mathcal{X}$ for which there exists a finite number of elements $u_1, \ldots, u_k$ of G and a finite number of scalars $\lambda_1, \ldots, \lambda_k \in \mathcal{S}$, such that $x = \hat{\bigoplus}_{i=1,\ldots,k} \lambda_i u_i$. Finally, if $C \in \mathcal{S}^{n \times r}$, we will denote by $\operatorname{Im} C$ the subsemimodule of $\mathcal{S}^n$ generated by the columns of C.

Let $(\mathcal{S}, \oplus, \otimes)$ denote a semiring. By a *system with coefficients in* $\mathcal{S}$, or a *system over* $\mathcal{S}$, we mean a discrete-time linear dynamical system whose evolution is determined by a set of equations of the form

$$
x(k) = Ax(k-1) \oplus Bu(k) \ , \tag{1}
$$

where $A \in \mathcal{S}^{n \times n}$, $B \in \mathcal{S}^{n \times q}$, and $x(k) \in \mathcal{S}^{n \times 1}$, $u(k) \in \mathcal{S}^{q \times 1}$, $k = 1, 2, \ldots$ are the sequences of state and control vectors respectively.

We are interested in studying the following problem: Given a certain specification for the state space of system (1), which we suppose is given by a semimodule $\mathcal{K} \subset \mathcal{S}^n$, we want to compute the maximal set of initial states $\mathcal{K}^*$ for which there exists a sequence of control vectors which makes the state of system (1) stay in $\mathcal{K}$ forever, that is, such that $x(k) \in \mathcal{K}$ for all $k \geq 0$. Note that $\mathcal{K}^*$ is clearly a subsemimodule of $\mathcal{S}^n$. To treat this problem it is convenient to make the following definition.

Definition 1 *Given the matrices $A \in \mathcal{S}^{n \times n}$ and $B \in \mathcal{S}^{n \times q}$, we say that a semimodule $\mathcal{X} \subset \mathcal{S}^n$ is* (geometrically) (A, B)-invariant *if for all $x \in \mathcal{X}$ there exists $u \in \mathcal{S}^q$ such that $Ax \oplus Bu$ belongs to $\mathcal{X}$.*

This definition clearly implies that a semimodule $\mathcal{X} \subset \mathcal{S}^n$ is (geometrically) (A, B)-invariant if and only if for all $x \in \mathcal{X}$ there exists a sequence of control vectors such that the trajectory of the dynamical system (1), associated with this control sequence and the initial condition $x(0) = x$, is completely contained in $\mathcal{X}$.

The proof of the following lemma is omitted as it is identical to the case of linear dynamical systems over rings.

Lemma 1 *If $\mathcal{K} \subset \mathcal{S}^n$ is a semimodule, then $\mathcal{K}^*$ is the maximal (geometrically) (A, B)-invariant semimodule contained in $\mathcal{K}$.*

To tackle the previous problem in the case of max-plus type semirings, we generalize the classical fixed point algorithm which is used to compute the maximal (A, B)-invariant subspace contained in a given space (see [Won85]). With this purpose in mind, we set $\mathcal{B} = \operatorname{Im} B$ and consider the self-map φ of the set of subsemimodules of $\mathcal{S}^n$, given by:

$$\varphi(\mathcal{X}) = \mathcal{X} \cap A^{-1}(\mathcal{X} \ominus \mathcal{B}) \ , \tag{2}$$

where the operation $\ominus$ is defined as follows:

$$\mathcal{Z} \ominus \mathcal{Y} = \{u \in \mathcal{S}^n \mid \exists y \in \mathcal{Y}, u \oplus y \in \mathcal{Z}\}, \quad \forall \mathcal{Z}, \mathcal{Y} \subset \mathcal{S}^n \ .$$

Note that when $\mathcal{S} = \mathbb{Z}_{\max}$ or $\mathcal{S} = \mathbb{N}_{\min}$, if the semimodule $\mathcal{X}$ is finitely generated, then the semimodule $\varphi(\mathcal{X})$ is also finitely generated and can be computed using a general elimination algorithm due to Butkovič and Hegedüs [BH84] and Gaubert [Gau92]. More generally, if $\mathcal{X}$ belongs to the class of rational semimodules (this class, which extends the notion of finitely generated semimodule, turns out to be useful in the geometric approach to discrete event systems, see [GK04]), then $\varphi(\mathcal{X})$ is also a rational semimodule and can be computed by Theorem 3.5 of [GK04].

Lemma 2 *A semimodule $\mathcal{X} \subset \mathcal{S}^n$ is (geometrically) (A, B)-invariant if and only if $\mathcal{X} = \varphi(\mathcal{X})$.*

Proof. Since

$$\begin{aligned} A^{-1}(\mathcal{X} \ominus \mathcal{B}) &= \{x \in \mathcal{S}^n \mid Ax \in \mathcal{X} \ominus \mathcal{B}\} = \\ &= \{x \in \mathcal{S}^n \mid \exists b \in \mathcal{B}, Ax \oplus b \in \mathcal{X}\} = \\ &= \{x \in \mathcal{S}^n \mid \exists u \in \mathcal{S}^q, Ax \oplus Bu \in \mathcal{X}\} \ , \end{aligned}$$

we can see that $A^{-1}(\mathcal{X} \ominus \mathcal{B})$ is the set of initial conditions $x(0)$ of the dynamical system (1) for which there exists a control $u(1)$ which makes the new state of the system, that is $x(1) = Ax(0) \oplus Bu(1)$, belong to $\mathcal{X}$. Then, it readily follows from Definition 1 that a semimodule $\mathcal{X} \subset \mathcal{S}^n$ is (geometrically) (A, B)-invariant if and only if $\mathcal{X} \subset A^{-1}(\mathcal{X} \ominus \mathcal{B})$. Therefore, a semimodule $\mathcal{X} \subset \mathcal{S}^n$ is (geometrically) (A, B)-invariant if and only if $\mathcal{X} = \varphi(\mathcal{X})$, that is, (geometrically) (A, B)-invariant semimodules are precisely the fixed points of the map φ defined by (2). ■

Inspired by the algorithm in the classical case, we define the following sequence of semimodules:

$$\mathcal{X}_1 = \mathcal{K} \ , \quad \mathcal{X}_{r+1} = \varphi(\mathcal{X}_r) \ , \quad \forall r \in \mathbb{N}. \tag{3}$$

Then we have the following lemma.

Lemma 3 *Let $\mathcal{K} \subset \mathcal{S}^n$ be an arbitrary semimodule. Then the sequence of semimodules $\{\mathcal{X}_r\}_{r \in \mathbb{N}}$ defined by (3) is decreasing, i.e. $\mathcal{X}_{r+1} \subset \mathcal{X}_r$ for all $r \in \mathbb{N}$. Moreover, if we define $\mathcal{X}_\omega = \cap_{r \in \mathbb{N}} \mathcal{X}_r$, then every (geometrically) (A, B)-invariant semimodule contained in $\mathcal{K}$ is also contained in $\mathcal{X}_\omega$. In particular, it follows that $\mathcal{K}^* \subset \mathcal{X}_\omega$.*

Proof. The fact that the sequence of semimodules $\{\mathcal{X}_r\}_{r\in\mathbb{N}}$ is decreasing is a straightforward consequence of the definition of the map φ:

$$\mathcal{X}_{r+1} = \varphi(\mathcal{X}_r) = \mathcal{X}_r \cap A^{-1}(\mathcal{X}_r \ominus \mathcal{B}) \subset \mathcal{X}_r,$$

for all $r \in \mathbb{N}$.

To prove the second part of Lemma 3, firstly it is convenient to notice that φ satisfies the following property:

$$\forall \mathcal{Z}, \mathcal{Y} \subset \mathcal{S}^n \ , \ \ \mathcal{Z} \subset \mathcal{Y} \Rightarrow \varphi(\mathcal{Z}) \subset \varphi(\mathcal{Y}) \ ,$$

that is, φ is monotonic when the set of subsemimodules of $\mathcal{S}^n$ is equipped with the order: $\mathcal{Z} \leq \mathcal{Y}$ if and only if $\mathcal{Z} \subset \mathcal{Y}$.

Now let $\mathcal{X} \subset \mathcal{K}$ be an arbitrary (geometrically) (A, B)-invariant semimodule. We will prove by induction on r that $\mathcal{X} \subset \mathcal{X}_r$ for all $r \in \mathbb{N}$, and therefore that $\mathcal{X} \subset \cap_{r\in\mathbb{N}}\mathcal{X}_r = \mathcal{X}_\omega$. In the first place, we know that $\mathcal{X} \subset \mathcal{K} = \mathcal{X}_1$. Since $\mathcal{X}$ is a (geometrically) (A, B)-invariant semimodule, thanks to Lemma 2, it follows that $\mathcal{X} = \varphi(\mathcal{X})$. If we now assume that $\mathcal{X} \subset \mathcal{X}_t$, then we have:

$$\mathcal{X} = \varphi(\mathcal{X}) \subset \varphi(\mathcal{X}_t) = \mathcal{X}_{t+1} \ .$$

Therefore, $\mathcal{X} \subset \mathcal{X}_r$ for all $r \in \mathbb{N}$, as we wanted to show. ∎

Note that if the sequence $\{\mathcal{X}_r\}_{r\in\mathbb{N}}$ stabilizes, that is, if there exists $k \in \mathbb{N}$ such that $\mathcal{X}_{k+1} = \mathcal{X}_k$, then our problem will be solved. Indeed, if there exists $k \in \mathbb{N}$ such that $\mathcal{X}_k = \mathcal{X}_{k+1} = \varphi(\mathcal{X}_k)$ then, thanks to Lemma 2, we know that $\mathcal{X}_k$ is a (geometrically) (A, B)-invariant semimodule which is contained in $\mathcal{K}$ (since $\mathcal{X}_1 = \mathcal{K}$ and by Lemma 3 the sequence $\{\mathcal{X}_r\}_{r\in\mathbb{N}}$ is decreasing). Therefore $\mathcal{X}_k \subset \mathcal{K}^*$, and as by Lemma 3 we know that $\mathcal{K}^* \subset \mathcal{X}_k$, it follows finally that $\mathcal{K}^* = \mathcal{X}_k$.

Example 1 *Let $\mathcal{S} = \mathbb{Z}_{\max}$. Let us consider the matrices*

$$A = \begin{pmatrix} -\infty & 0 \\ 0 & -\infty \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ ,$$

and the semimodule $\mathcal{K} = \{(x, y)^T \in \mathbb{Z}^2_{\max} \mid y \geq x + 1\}$. Let us compute, in this particular case, the sequence of semimodules $\{\mathcal{X}_r\}_{r\in\mathbb{N}}$ defined by (3). By definition we know that $\mathcal{X}_1 = \mathcal{K} = \{(x, y)^T \in \mathbb{Z}^2_{\max} \mid y \geq x + 1\}$. To compute $\mathcal{X}_2 = \varphi(\mathcal{X}_1) = \mathcal{X}_1 \cap A^{-1}(\mathcal{X}_1 \ominus \mathcal{B})$, first of all it is straightforward to see that $\mathcal{X}_1 \ominus \mathcal{B} = \mathcal{X}_1$. Therefore,

$$\begin{aligned} A^{-1}(\mathcal{X}_1 \ominus \mathcal{B}) &= A^{-1}(\mathcal{X}_1) = \\ &= \{(x, y)^T \in \mathbb{Z}^2_{\max} \mid A(x, y)^T \in \mathcal{X}_1\} = \\ &= \{(x, y)^T \in \mathbb{Z}^2_{\max} \mid (y, x)^T \in \mathcal{X}_1\} = \\ &= \{(x, y)^T \in \mathbb{Z}^2_{\max} \mid x \geq y + 1\} \ , \end{aligned}$$

and thus

$$\begin{aligned}\mathcal{X}_2 &= \mathcal{X}_1 \cap A^{-1}(\mathcal{X}_1 \ominus \mathcal{B}) = \\ &= \{(x,y)^T \in \mathbb{Z}^2_{\max} \mid y \geq x+1\} \cap \{(x,y)^T \in \mathbb{Z}^2_{\max} \mid x \geq y+1\} = \\ &= \{(-\infty,-\infty)^T\} \ .\end{aligned}$$

Then, since by Lemma 3 the sequence of semimodules $\{\mathcal{X}_r\}_{r\in\mathbb{N}}$ is decreasing, it follows that $\mathcal{X}_k = \{(-\infty,-\infty)^T\}$ for all $k \geq 2$. Therefore, the maximal (geometrically) (A,B)-invariant semimodule contained in $\mathcal{K}$ is trivial: $\mathcal{K}^* = \mathcal{X}_\omega = \{(-\infty,-\infty)^T\}$.

In the case of the theory of linear dynamical systems over a field, the sequence $\{\mathcal{X}_r\}_{r\in\mathbb{N}}$ always converges in at most n steps, since it is a decreasing sequence of subspaces of a vector space of dimension n. However, one of the problems in the max-plus case, which is reminiscent of difficulties of the theory of linear dynamical systems over rings (see [Ass99, ALP99, CP94, CP95, Hau82, Hau84]), is that the sequence $\{\mathcal{X}_r\}_{r\in\mathbb{N}}$ may not stabilize (see Example 2 below). This difficulty comes from the fact that the semimodule $\mathbb{Z}^n_{\max}$ is not Artinian, that is, there are infinite decreasing sequences of subsemimodules of $\mathbb{Z}^n_{\max}$. In the case of linear dynamical systems over rings, the convergence of the sequence $\{\mathcal{X}_r\}_{r\in\mathbb{N}}$ in a finite number of steps is not guaranteed either, and although there exists a procedure for finding $\mathcal{K}^*$ when $\mathcal{S}$ is a Principal Ideal Domain (see [CP94]), in general the computation of $\mathcal{K}^*$ remains a difficult problem.

Example 2 *Let $\mathcal{S} = \mathbb{Z}_{\max}$. Let us consider the matrices*

$$A = \begin{pmatrix} -1 & -\infty \\ -\infty & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 \\ 0 \end{pmatrix} ,$$

and the semimodule $\mathcal{K} = \{(x,y)^T \in \mathbb{Z}^2_{\max} \mid y \leq x-1\}$. Note that $\mathcal{K} = \operatorname{Im} K$, where

$$K = \begin{pmatrix} 0 & 0 \\ -1 & -\infty \end{pmatrix} .$$

Next we show that in this case the sequence of semimodules $\{\mathcal{X}_r\}_{r\in\mathbb{N}}$ defined by (3) is given by:

$$\mathcal{X}_r = \{(x,y)^T \in \mathbb{Z}^2_{\max} \mid y \leq x-r\} = \operatorname{Im} \begin{pmatrix} 0 & 0 \\ -r & -\infty \end{pmatrix} , \tag{4}$$

for all $r \in \mathbb{N}$. We prove equality (4) by induction on r. Let us note, in the first place, that equality (4) is satisfied by definition when $r = 1$. Assume now that equality (4) holds for $r = k$, that is:

$$\mathcal{X}_k = \{(x,y)^T \in \mathbb{Z}^2_{\max} \mid y \leq x-k\} = \operatorname{Im} \begin{pmatrix} 0 & 0 \\ -k & -\infty \end{pmatrix} .$$

Let us note that $\mathcal{X}_k \ominus \mathcal{B} = \mathcal{X}_k$*, since there exists* $\lambda \in \mathbb{Z}_{\max}$ *such that* $\max(y,\lambda) \leq \max(x,\lambda) - k$ *(that is, there exists* $(\lambda,\lambda)^T \in \mathcal{B}$ *such that* $(x,y)^T \oplus (\lambda,\lambda)^T \in \mathcal{X}_k$*) if and only if* $y \leq x - k$ *(that is,* $(x,y)^T \in \mathcal{X}_k$*). Therefore,*

$$\begin{aligned} A^{-1}(\mathcal{X}_k \ominus \mathcal{B}) &= A^{-1}(\mathcal{X}_k) = \\ &= \{(x,y)^T \in \mathbb{Z}^2_{\max} \mid A(x,y)^T \in \mathcal{X}_k\} = \\ &= \{(x,y)^T \in \mathbb{Z}^2_{\max} \mid (x-1,y)^T \in \mathcal{X}_k\} = \\ &= \{(x,y)^T \in \mathbb{Z}^2_{\max} \mid y \leq x - 1 - k\} \;, \end{aligned}$$

and thus

$$\begin{aligned} \mathcal{X}_{k+1} &= \mathcal{X}_k \cap A^{-1}(\mathcal{X}_k \ominus \mathcal{B}) = \\ &= \{(x,y)^T \in \mathbb{Z}^2_{\max} \mid y \leq x - k\} \cap \{(x,y)^T \in \mathbb{Z}^2_{\max} \mid y \leq x - 1 - k\} = \\ &= \{(x,y)^T \in \mathbb{Z}^2_{\max} \mid y \leq x - (1+k)\} \;, \end{aligned}$$

which shows that equality (4) holds for all $r \in \mathbb{N}$.

We see in this way that the sequence of semimodules $\{\mathcal{X}_r\}_{r\in\mathbb{N}}$ *is strictly decreasing and therefore does not stabilize. Let us finally note that the semimodule* $\mathcal{X}_\omega = \cap_{r\in\mathbb{N}}\mathcal{X}_r = \{(x,y)^T \in \mathbb{Z}^2_{\max} \mid y = -\infty\}$ *is* A*-invariant, that is,* $A(\mathcal{X}_\omega) \subset \mathcal{X}_\omega$*. Then,* $\mathcal{X}_\omega$ *is in particular (geometrically)* (A,B)*-invariant and therefore* $\mathcal{K}^* = \mathcal{X}_\omega = \{(x,y)^T \in \mathbb{Z}^2_{\max} \mid y = -\infty\}$.

An open problem is to determine whether it is always the case that $\mathcal{K}^* = \mathcal{X}_\omega$. It is worth mentioning that this equality does not necessarily hold in the case of linear dynamical systems over rings. Even when $\mathcal{S}$ is a Principal Ideal Domain, it could be necessary to compute more than once (but a finite number of times) the limit $\mathcal{X}_\omega$ of sequences defined as in (3). To be more precise, in such a case $\mathcal{X}_1$ is defined as $\mathcal{K}$ in the first step and, if it is necessary (that is, when $\mathcal{X}_\omega$ is not a geometrically (A,B)-invariant module), in the next steps $\mathcal{X}_1$ is defined as the smallest *closed* submodule containing the previous limit $\mathcal{X}_\omega$ (see [CP94] for details). Sufficient conditions for the stabilization of the sequence $\{\mathcal{X}_r\}_{r\in\mathbb{N}}$ defined by (3), and therefore for the equality $\mathcal{K}^* = \mathcal{X}_\omega$ to hold true, will be given in Section 4 in the case of $\mathcal{S} = \mathbb{Z}_{\max}$. Note that Example 2 shows that even in the case of the tropical semiring $\mathbb{N}_{\min}$ the sequence of semimodules $\{\mathcal{X}_r\}_{r\in\mathbb{N}}$ may not stabilize (indeed all the computations in Example 2 are valid when we restrict ourselves to the semiring $\mathbb{N}^-_{\max} = (\mathbb{N}^- \cup \{-\infty\}, \max, +)$, which is clearly isomorphic to $\mathbb{N}_{\min}$). However, more general sufficient conditions for the equality $\mathcal{K}^* = \mathcal{X}_\omega$ to hold true can be given in the case of the tropical semiring using compactness arguments. With this aim, let us consider the topology of $\mathbb{N}_{\min}$ defined by the metric:

$$d(x,y) = |\exp(-x) - \exp(-y)| \;,$$

for all $x, y \in \mathbb{N}_{\min}$. Then we have the following lemma.

Lemma 4 *Finitely generated subsemimodules of* $\mathbb{N}^n_{\min}$ *are compact.*

Proof. In the first place, let us notice that $\mathbb{N}_{\min}$ is a *topological semiring*, that is, for all sequences $\{x_r\}_{r\in\mathbb{N}}$ and $\{y_r\}_{r\in\mathbb{N}}$ of elements of $\mathbb{N}_{\min}$ the following equalities are satisfied:

$$\lim_{r\to\infty}(x_r \oplus y_r) = \left(\lim_{r\to\infty} x_r\right) \oplus \left(\lim_{r\to\infty} y_r\right) ,$$

and

$$\lim_{r\to\infty}(x_r \otimes y_r) = \left(\lim_{r\to\infty} x_r\right) \otimes \left(\lim_{r\to\infty} y_r\right) .$$

Let us now see that every finitely generated semimodule $\mathcal{X} \subset \mathbb{N}_{\min}^n$ is compact. Indeed, since $\mathcal{X}$ is finitely generated there exists a matrix $Q \in \mathbb{N}_{\min}^{n\times p}$, for some $p \in \mathbb{N}$, such that $\mathcal{X} = \operatorname{Im} Q$. Let $\{x_r\}_{r\in\mathbb{N}}$ be an arbitrary sequence of elements of $\mathcal{X}$. To prove that $\mathcal{X}$ is compact, we must show that $\{x_r\}_{r\in\mathbb{N}}$ has a subsequence which converges to an element of $\mathcal{X}$. Since $\{x_r\}_{r\in\mathbb{N}} \subset \mathcal{X}$, there exists a sequence $\{y_r\}_{r\in\mathbb{N}} \subset \mathbb{N}_{\min}^p$ such that $x_r = Qy_r$ for all $r \in \mathbb{N}$. Now it is straightforward to see that there exists a subsequence $\{y_{r_k}\}_{k\in\mathbb{N}}$ of $\{y_r\}_{r\in\mathbb{N}}$ and an element $y \in \mathbb{N}_{\min}^p$ such that $\lim_{k\to\infty} y_{r_k} = y$. Then, using the fact that $\mathbb{N}_{\min}$ is a topological semiring, it follows that

$$\lim_{k\to\infty} x_{r_k} = \lim_{k\to\infty}(Qy_{r_k}) = Q\left(\lim_{k\to\infty} y_{r_k}\right) = Qy \in \mathcal{X} .$$

Therefore, $\mathcal{X}$ is compact. ■

The following theorem shows that in the case of $\mathbb{N}_{\min}$ the equality $\mathcal{K}^* = \mathcal{X}_\omega$ holds when $\mathcal{K}$ is finitely generated.

Theorem 1 *Let $\mathcal{K} \subset \mathbb{N}_{\min}^n$ be a finitely generated semimodule. Then, for all matrices $A \in \mathbb{N}_{\min}^{n\times n}$ and $B \in \mathbb{N}_{\min}^{n\times q}$, the maximal (geometrically) (A,B)-invariant semimodule $\mathcal{K}^*$ contained in $\mathcal{K}$ is given by $\mathcal{X}_\omega = \cap_{r\in\mathbb{N}}\mathcal{X}_r$, where the sequence of semimodules $\{\mathcal{X}_r\}_{r\in\mathbb{N}}$ is defined by (3).*

Proof. By Lemma 3, to prove the theorem, it suffices to show that $\mathcal{X}_\omega$ is a (geometrically) (A,B)-invariant semimodule, which is equivalent to showing that $\mathcal{X}_\omega = \varphi(\mathcal{X}_\omega)$ by Lemma 2.

Since $\mathcal{X}_\omega \subset \mathcal{X}_r$ for all $r \in \mathbb{N}$, it follows that $\varphi(\mathcal{X}_\omega) \subset \varphi(\mathcal{X}_r) = \mathcal{X}_{r+1}$ for all $r \in \mathbb{N}$. Therefore, $\varphi(\mathcal{X}_\omega) \subset \cap_{r\in\mathbb{N}}\mathcal{X}_r = \mathcal{X}_\omega$.

Let us now see that $\mathcal{X}_\omega \subset \varphi(\mathcal{X}_\omega)$. Let x be an arbitrary element of $\mathcal{X}_\omega$. Then, since $x \in \varphi(\mathcal{X}_r) = \mathcal{X}_{r+1}$ for all $r \in \mathbb{N}$, we know that there exists a sequence $\{b_r\}_{r\in\mathbb{N}} \subset \mathcal{B}$ such that $Ax \oplus b_r$ belongs to $\mathcal{X}_r$ for all $r \in \mathbb{N}$. As $\mathcal{B}$ is compact by Lemma 4, there exists $b \in \mathcal{B}$ and a subsequence $\{b_{r_k}\}_{k\in\mathbb{N}}$ of $\{b_r\}_{r\in\mathbb{N}}$ such that $\lim_{k\to\infty} b_{r_k} = b$. Now, since by Lemma 3 the sequence of semimodules $\{\mathcal{X}_r\}_{r\in\mathbb{N}}$ is decreasing, it follows that $Ax \oplus b_{r_j} \in \mathcal{X}_{r_k}$ for all $j \geq k$. Therefore, $Ax \oplus b \in \mathcal{X}_{r_k}$ for all $k \in \mathbb{N}$ (recall that the semimodules $\mathcal{X}_r$ are all finitely generated and then, by Lemma 4, in particular closed). Then, $Ax \oplus b$ belongs to $\mathcal{X}_\omega$, from which we can see that $x \in \varphi(\mathcal{X}_\omega)$. Therefore, $\mathcal{X}_\omega \subset \varphi(\mathcal{X}_\omega)$. ■

3 Volume

In the next section we will give sufficient conditions on the semimodule $\mathcal{K}$, when $\mathcal{S} = \mathbb{Z}_{\max}$, to assure that the sequence of semimodules $\{\mathcal{X}_r\}_{r\in\mathbb{N}}$ defined by (3) stabilizes. For this purpose it is convenient to introduce first the notion of volume of a subsemimodule of $\mathbb{Z}_{\max}^n$ and study its properties.

Definition 2 *Let $\mathcal{K} \subset \mathbb{Z}_{\max}^n$ be a semimodule. We call the* volume *of $\mathcal{K}$, represented by $\mathrm{vol}\,(\mathcal{K})$, the cardinality of the set $\{x \in \mathcal{K} \mid x_1 \oplus \cdots \oplus x_n = 0\}$, that is, $\mathrm{vol}\,(\mathcal{K}) = \mathrm{card}\,(\{x \in \mathcal{K} \mid x_1 \oplus \cdots \oplus x_n = 0\})$. Also, if $K \in \mathbb{Z}_{\max}^{n\times p}$, we represent by $\mathrm{vol}\,(K)$ the volume of the semimodule $\mathcal{K} = \mathrm{Im}\, K$, that is, $\mathrm{vol}\,(K) = \mathrm{vol}\,(\mathrm{Im}\, K)$.*

Remark 1 *Let us consider the max-plus parallelism relation $\sim$ on $\mathbb{Z}_{\max}^n$ defined by: $x \sim y$ if and only if $x = \lambda y$ for some $\lambda \in \mathbb{R}$ (that is, $x_i = \lambda + y_i$ for all $1 \leq i \leq n$, in the usual algebra). Then, it is straightforward to see that the volume of a semimodule $\mathcal{K} \subset \mathbb{Z}_{\max}^n$ is equal to $\mathrm{card}\,(\mathcal{K}/\sim) - 1$, that is, the cardinality of the set of* nontrivial lines *contained in $\mathcal{K}$ (here $\mathcal{K}/\sim$ denotes the quotient of $\mathcal{K}$ by the parallelism relation $\sim$). The* max-plus projective space *is the quotient of $\mathbb{R}_{\max}^n$ by the parallelism relation.*

Before stating the following lemma, which provides some properties of the volume, it is convenient to introduce the following notation: if $\mathcal{X} \subset \mathbb{Z}_{\max}^n$, then we define

$$\tilde{\mathcal{X}} = \{x \in \mathcal{X} \mid x_1 \oplus \cdots \oplus x_n = 0\} .$$

Lemma 5 *Let $A \in \mathbb{Z}_{\max}^{r\times n}$, $B \in \mathbb{Z}_{\max}^{n\times p}$ and $C \in \mathbb{Z}_{\max}^{p\times q}$ be matrices and $\mathcal{Z}, \mathcal{Y} \subset \mathbb{Z}_{\max}^n$ be semimodules. Then we have:*

1. $\mathcal{Y} \subset \mathcal{Z} \Rightarrow \mathrm{vol}\,(\mathcal{Y}) \leq \mathrm{vol}\,(\mathcal{Z})$,
2. *if* $\mathrm{vol}\,(\mathcal{Y}) < \infty$, *then* $\mathcal{Y} \subsetneq \mathcal{Z} \Rightarrow \mathrm{vol}\,(\mathcal{Y}) < \mathrm{vol}\,(\mathcal{Z})$,
3. $\mathrm{vol}\,(A\mathcal{Y}) \leq \mathrm{vol}\,(A)$ *and then* $\mathrm{vol}\,(AB) \leq \mathrm{vol}\,(A)$,
4. $\mathrm{vol}\,(A\mathcal{Y}) \leq \mathrm{vol}\,(\mathcal{Y})$ *and then* $\mathrm{vol}\,(AB) \leq \mathrm{vol}\,(B)$,
5. $\mathrm{vol}\,(ABC) \leq \mathrm{vol}\,(B)$,
6. *if* $P \in \mathbb{Z}_{\max}^{n\times n}$ *and* $Q \in \mathbb{Z}_{\max}^{p\times p}$ *are invertible, then* $\mathrm{vol}\,(PBQ) = \mathrm{vol}\,(B)$,
7. $\mathrm{vol}\,(A) = \mathrm{vol}\,(A^T)$.

Proof. 1. This property is a straightforward consequence of the definition of volume: $\mathcal{Y} \subset \mathcal{Z} \Rightarrow \tilde{\mathcal{Y}} \subset \tilde{\mathcal{Z}} \Rightarrow \mathrm{card}\,(\tilde{\mathcal{Y}}) \leq \mathrm{card}\,(\tilde{\mathcal{Z}}) \Rightarrow \mathrm{vol}\,(\mathcal{Y}) \leq \mathrm{vol}\,(\mathcal{Z})$.

2. In the first place, we will show that the following simple property is satisfied: for all semimodules $\mathcal{Y}, \mathcal{Z} \subset \mathbb{Z}_{\max}^n$,

$$\mathcal{Y} \subsetneq \mathcal{Z} \Rightarrow \tilde{\mathcal{Y}} \subsetneq \tilde{\mathcal{Z}} . \tag{5}$$

In effect, assume that $\mathcal{Y} \subsetneq \mathcal{Z}$. Then there exists $x \in \mathcal{Z} \setminus \mathcal{Y}$. Therefore, we know that $x \neq (-\infty, \ldots, -\infty)^T$ and we can define the vector $\tilde{x} = (x_1 \oplus \cdots \oplus x_n)^{-1} x$ (that is, $\tilde{x}_i = x_i - \max\{x_1, \ldots, x_n\}$ for all $1 \leq i \leq n$, in the usual algebra). Now, it is easy to check that $\tilde{x} \in \tilde{\mathcal{Z}} \setminus \tilde{\mathcal{Y}}$ and thus $\tilde{\mathcal{Y}} \subsetneq \tilde{\mathcal{Z}}$. This proves property (5).

Now, using property (5) and the fact that $\mathrm{vol}\,(\mathcal{Y}) < \infty$, we get: $\mathcal{Y} \subsetneq \mathcal{Z} \Rightarrow \tilde{\mathcal{Y}} \subsetneq \tilde{\mathcal{Z}} \Rightarrow \mathrm{card}\,(\tilde{\mathcal{Y}}) < \mathrm{card}\,(\tilde{\mathcal{Z}}) \Rightarrow \mathrm{vol}\,(\mathcal{Y}) < \mathrm{vol}\,(\mathcal{Z})$.

3. Since $A\mathcal{Y} \subset \mathrm{Im}\,A$, applying Statement 1, we have: $\mathrm{vol}\,(A\mathcal{Y}) \leq \mathrm{vol}\,(\mathrm{Im}\,A) = \mathrm{vol}\,(A)$.

4. From the definition of the set $\tilde{\mathcal{Y}}$ it is easy to see that for all $y \in \mathcal{Y} \setminus \{(-\infty, \ldots, -\infty)^T\}$ there exists $\tilde{y} \in \tilde{\mathcal{Y}}$ and $\lambda \in \mathbb{Z}$ such that $y = \lambda \tilde{y}$ (it suffices to take $\lambda = y_1 \oplus \cdots \oplus y_n$ and $\tilde{y} = \lambda^{-1} y$). Therefore,

$$A\mathcal{Y} \setminus \{(-\infty, \ldots, -\infty)^T\} \subset \{\lambda A\tilde{y} \mid \tilde{y} \in \tilde{\mathcal{Y}}, \lambda \in \mathbb{Z}\}\,,$$

and then we get:

$$\begin{aligned}
\mathrm{vol}\,(A\mathcal{Y}) &= \mathrm{card}\,(\{x \in A\mathcal{Y} \mid x_1 \oplus \cdots \oplus x_r = 0\}) \\
&\leq \mathrm{card}\,(\{x = \lambda A\tilde{y} \mid \tilde{y} \in \tilde{\mathcal{Y}}, \lambda \in \mathbb{Z}, x_1 \oplus \cdots \oplus x_r = 0\}) \\
&\leq \mathrm{card}\,(\{A\tilde{y} \mid \tilde{y} \in \tilde{\mathcal{Y}}\}) \leq \mathrm{card}\,(\tilde{\mathcal{Y}}) = \mathrm{vol}\,(\mathcal{Y})\,.
\end{aligned}$$

5. Applying Statements 3 and 4 we get: $\mathrm{vol}\,(ABC) \leq \mathrm{vol}\,(AB) \leq \mathrm{vol}\,(B)$.

6. From Statement 5 we obtain: $\mathrm{vol}\,(B) = \mathrm{vol}\,(P^{-1}PBQQ^{-1}) \leq \mathrm{vol}\,(PBQ) \leq \mathrm{vol}\,(B)$. Therefore, $\mathrm{vol}\,(B) = \mathrm{vol}\,(PBQ)$.

7. Let us note, in the first place, that we can define in a completely analogous way the volume of a subsemimodule of $\mathbb{Z}^n_{\min}$. Then, since the function $x \to -x$ is an isomorphism from $\mathbb{Z}_{\max}$ to $\mathbb{Z}_{\min}$, it is clear that $\mathrm{vol}\,(\mathcal{Z}) = \mathrm{vol}\,(-\mathcal{Z})$ for every subsemimodule $\mathcal{Z} \subset \mathbb{Z}^n_{\max}$. Let us now consider the matrix $A^\sharp = -A^T$ and the semimodule $\mathcal{Y} = \mathrm{Im}\,(A^\sharp) \subset \mathbb{Z}^n_{\min}$. Since $\mathcal{Y} = -\mathrm{Im}\,(A^T)$, we know that $\mathrm{vol}\,(A^T) = \mathrm{vol}\,(\mathcal{Y})$. Now, using elements of residuation theory, it can be shown (see for example [BCOQ92] or [CGQ01]) that the following two properties hold:

$$\begin{aligned}
A(A^\sharp(Ax)) &= Ax\,,\ \ \forall x \in \mathbb{Z}^n_{\max}\,,\ \text{and} \\
A^\sharp(A(A^\sharp y)) &= A^\sharp y\,,\ \ \forall y \in \mathbb{Z}^r_{\min}\,,
\end{aligned}$$

where the products by A are performed in $\overline{\mathbb{Z}}_{\max}$ and the products by $A^\sharp$ are performed in $\overline{\mathbb{Z}}_{\min}$. Therefore, the function $f : \mathrm{Im}\,(A) \mapsto \mathrm{Im}\,(A^\sharp)$ defined by $f(y) = A^\sharp y$ is a bijection with inverse $g(x) = Ax$. Then, the function F from $\mathrm{Im}\,(A)/\sim$ to $\mathrm{Im}\,(A^\sharp)/\sim$ defined by $F([y]) = [A^\sharp y]$, where $[x]$ denotes the equivalence class of x by the parallelism relation $\sim$, is also a bijection. Now, using Remark 1, we obtain: $\mathrm{vol}\,(A) = \mathrm{card}\,(\mathrm{Im}\,(A)/\sim) - 1 = \mathrm{card}\,(\mathrm{Im}\,(A^\sharp)/\sim) - 1 = \mathrm{vol}\,(A^\sharp) = \mathrm{vol}\,(\mathcal{Y})$, and then $\mathrm{vol}\,(A) = \mathrm{vol}\,(\mathcal{Y}) = \mathrm{vol}\,(A^T)$. ■

4 Specifications with finite volume

In the next theorem we give a condition on the specification $\mathcal{K}$, when $\mathcal{S} = \mathbb{Z}_{\max}$, ensuring that the sequence of semimodules defined by (3) stabilizes.

Theorem 2 *Let $\mathcal{K} \subset \mathbb{Z}^n_{\max}$ be a semimodule such that* $\mathrm{vol}\,(\mathcal{K}) < \infty$*, that is, a semimodule with finite volume. Then, for all $A \in \mathbb{Z}^{n\times n}_{\max}$ and $B \in \mathbb{Z}^{n\times p}_{\max}$, the maximal (geometrically) (A, B)-invariant semimodule $\mathcal{K}^*$ contained in $\mathcal{K}$ is finitely generated. Moreover, if we define the sequence of semimodules $\{\mathcal{X}_r\}_{r\in\mathbb{N}}$ by (3), then $\mathcal{K}^* = \mathcal{X}_k$ for some $k \leq \mathrm{vol}\,(\mathcal{K}) + 1$.*

Proof. First of all, let us note that every semimodule $\mathcal{Y} \subset \mathbb{Z}^n_{\max}$ with finite volume is necessarily finitely generated. Indeed, this property is a straightforward consequence of the fact that $\mathcal{Y} = \mathrm{span}\,(\tilde{\mathcal{Y}})$. Now, as $\mathcal{K}^* \subset \mathcal{K}$, applying Statement 1 of Lemma 5 it follows that $\mathrm{vol}\,(\mathcal{K}^*) \leq \mathrm{vol}\,(\mathcal{K}) < \infty$, and then $\mathcal{K}^*$ is finitely generated.

Let us now see that the sequence of semimodules $\{\mathcal{X}_r\}_{r\in\mathbb{N}}$ defined by (3) must stabilize in at most $\mathrm{vol}\,(\mathcal{K})+1$ steps. Indeed, by Lemma 3 we know that the sequence of semimodules $\{\mathcal{X}_r\}_{r\in\mathbb{N}}$ is decreasing. Then, using Statement 1 of Lemma 5, we can see that $\{\mathrm{vol}\,(\mathcal{X}_r)\}_{r\in\mathbb{N}}$ is a decreasing sequence of nonnegative integers. Therefore, there exists $k \leq \mathrm{vol}\,(\mathcal{X}_1) + 1 = \mathrm{vol}\,(\mathcal{K}) + 1$ such that $\mathrm{vol}\,(\mathcal{X}_{k+1}) = \mathrm{vol}\,(\mathcal{X}_k)$. Then, as $\mathcal{X}_{k+1} \subset \mathcal{X}_k \subset \mathcal{K}$ by Lemma 3, we know that $\mathrm{vol}\,(\mathcal{X}_{k+1}) = \mathrm{vol}\,(\mathcal{X}_k) \leq \mathrm{vol}\,(\mathcal{K}) < \infty$ (once again, by Statement 1 of Lemma 5). Finally, applying Statement 2 of Lemma 5 to the semimodules $\mathcal{X}_{k+1}$ and $\mathcal{X}_k$, it follows that $\mathcal{X}_{k+1} = \mathcal{X}_k$, from which we conclude that $\mathcal{K}^* = \mathcal{X}_k$. ■

An important particular case of Theorem 2 is that where the semimodule $\mathcal{K}$ is generated by a finite number of vectors whose entries are all finite. In this case it is possible to bound the volume of $\mathcal{K}$ by means of the additive version of Hilbert's projective metric: for all $x \in \mathbb{Z}^n$, define

$$\|x\|_H = \max\{x_i \mid 1 \leq i \leq n\} - \min\{x_i \mid 1 \leq i \leq n\} \ ,$$

and for all $K \in \mathbb{Z}^{n\times s}$, define

$$\Delta_H(K) = \max\{\|K_{\cdot i}\|_H \mid 1 \leq i \leq s\} \ ,$$

where $K_{\cdot i}$ denotes the i-th column of the matrix K. Then we have the following corollary.

Corollary 1 *Let $\mathcal{K} = \mathrm{Im}\,K$, where $K \in \mathbb{Z}^{n\times s}_{\max}$ is a matrix whose entries are all finite. Then, for all $A \in \mathbb{Z}^{n\times n}_{\max}$ and $B \in \mathbb{Z}^{n\times p}_{\max}$, the maximal (geometrically) (A, B)-invariant semimodule $\mathcal{K}^*$ contained in $\mathcal{K}$ is finitely generated and, if we define the sequence of semimodules $\{\mathcal{X}_r\}_{r\in\mathbb{N}}$ by (3), there exists some $k \leq (\Delta_H(K) + 1)^n - \Delta_H(K)^n + 1$ such that $\mathcal{K}^* = \mathcal{X}_k$.*

Proof. By Theorem 2, to prove the corollary, it suffices to show that

$$\mathrm{vol}\,(\mathcal{K}) \leq (\Delta_H(K) + 1)^n - \Delta_H(K)^n \ , \tag{6}$$

where the power n is in the usual algebra.

Since the additive version of Hilbert's projective metric $\|\cdot\|_H$ satisfies the following straightforward properties:

$$\begin{aligned} \|\lambda x\|_H &= \|x\|_H \ , \\ \|x \oplus y\|_H &\leq \|x\|_H \oplus \|y\|_H \ , \end{aligned}$$

for all $x, y \in \mathbb{Z}^n$ and $\lambda \in \mathbb{Z}$, it follows that $\|x\|_H \leq \Delta_H(K)$ for all $x \in \mathcal{K}\setminus\{(-\infty,\ldots,-\infty)^T\}$ and therefore $\mathcal{K}$ is contained in the semimodule

$$\mathcal{Y} = \{x \in \mathbb{Z}^n \mid \|x\|_H \leq \Delta_H(K)\} \cup \{(-\infty,\ldots,-\infty)^T\}$$

(note that the only vector in $\mathcal{K}$ with at least one coordinate equal to $-\infty$ is $(-\infty,\ldots,-\infty)^T$). Then, by Statement 1 of Lemma 5, to prove inequality (6) it suffices to show that $\mathrm{vol}\,(\mathcal{Y}) = (\Delta_H(K)+1)^n - \Delta_H(K)^n$. With this aim, we must compute the number of elements of the set:

$$\begin{aligned} \tilde{\mathcal{Y}} &= \{x \in \mathcal{Y} \mid x_1 \oplus \cdots \oplus x_n = 0\} = \\ &= \{x \in \mathbb{Z}^n \mid \|x\|_H \leq \Delta_H(K), x_1 \oplus \cdots \oplus x_n = 0\}\ , \end{aligned}$$

that is, the number of vectors x of $\mathbb{Z}^n$ with at least one coordinate equal to zero (since $\max_i x_i = x_1 \oplus \cdots \oplus x_n = 0$) and with the rest of the coordinates greater than or equal to $-\Delta_H(K)$ (since $\Delta_H(K) \geq \|x\|_H = \max_i x_i - \min_i x_i = -\min_i x_i$). We know that there are $\binom{n}{r}\Delta_H(K)^{n-r}$ elements in the set $\tilde{\mathcal{Y}}$ with exactly r coordinates equal to zero. To be more precise, there exist $\binom{n}{r}$ different ways of choosing the r coordinates which will have the value zero, and there exist $\Delta_H(K)^{n-r}$ different ways of assigning values to the $n-r$ remaining coordinates among the $\Delta_H(K)$ possible values. Therefore, the number of elements of the set $\tilde{\mathcal{Y}}$ is:

$$\sum_{r=1}^{r=n} \binom{n}{r} \Delta_H(K)^{n-r} = (\Delta_H(K)+1)^n - \Delta_H(K)^n\ ,$$

and then $\mathrm{vol}\,(\mathcal{Y}) = (\Delta_H(K)+1)^n - \Delta_H(K)^n$. ■

Note that in the proof of Corollary 1 we showed, in particular, that for each matrix $K \in \mathbb{Z}_{\max}^{n\times s}$ whose entries are all finite, the volume $\mathrm{vol}\,(K)$ is bounded by $(\Delta_H(K)+1)^n - \Delta_H(K)^n$ (this is inequality (6)). We next show that this bound is tight. Indeed, let us consider the semimodule

$$\mathcal{Y} = \{x \in \mathbb{Z}^n \mid \|x\|_H \leq M\} \cup \{(-\infty,\ldots,-\infty)^T\}\ ,$$

where $M \in \mathbb{N}$. Note that in the proof of Corollary 1 we proved that $\mathcal{Y}$ has volume $(M+1)^n - M^n$. Now, if we define the matrix

$$K = \begin{pmatrix} M & 0 & \cdots & 0 & 0 \\ 0 & M & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & M & 0 \\ 0 & 0 & \cdots & 0 & M \end{pmatrix} \in \mathbb{Z}_{\max}^{n\times n}\ ,$$

it is straightforward to verify that $\mathcal{Y} = \mathrm{Im}\,(K)$ and $\Delta_H(K) = M$. Therefore, there exist matrices $K \in \mathbb{Z}_{\max}^{n\times s}$ (whose entries are all finite) which have volume equal to $(\Delta_H(K)+1)^n - \Delta_H(K)^n$.

Theorem 2 is useful in many practical problems because in such problems the specification $\mathcal{K}$ frequently has finite volume. This is often the case when $\mathcal{K}$ models certain stability conditions, as for example, "bounded delay" requirements. To be more precise, let us assume that system (1) is the dater representation of a timed event graph (we refer the reader to [BCOQ92] for more details on the modeling of timed event graphs). Then, a typical case of semimodule $\mathcal{K}$ which arises in applications is:

$$\mathcal{K} = \{x \in \mathbb{Z}_{\max}^n \mid x_i - x_j \leq d_{ij}, \forall 1 \leq i, j \leq n\} , \tag{7}$$

where $D = (d_{ij})$ is a matrix with entries in $\mathbb{Z} \cup \{+\infty\}$. Note that the state vector for the firings number k, $x(k)$, belongs to $\mathcal{K}$ if and only if $x(k)_i - x(k)_j \leq d_{ij}$, for all $1 \leq i, j \leq n$, which means that the delay between the k-th firing of the transition labeled j and the k-th firing of the transition labeled i should not exceed d_{ij}. We next show that the semimodule $\mathcal{K}$ defined by (7) often has finite volume. Let us first recall that to any matrix $A \in \mathbb{R}_{\max}^{n\times n}$ is associated a directed graph $\mathcal{G}(A)$, called the *precedence graph* of A, which is defined as follows: there exists an oriented arc of *weight* A_{ji} from node i to node j if and only if $A_{ji} \neq -\infty$. A matrix whose precedence graph is strongly connected is called *irreducible.* The spectral radius $\rho_{\max}(A)$ of A is defined by:

$$\rho_{\max}(A) = \bigoplus_{k=1}^{n} \operatorname{tr}(A^k)^{\frac{1}{k}} = \max_{1\leq k\leq n} \max_{i_1,\dots,i_k} \frac{A_{i_1 i_2} + \cdots + A_{i_k i_1}}{k} ,$$

that is, the maximal circuit mean of $\mathcal{G}(A)$.

To see that the semimodule (7) often has finite volume, let us first note that

$$\mathcal{K} = \{x \in \mathbb{Z}_{\max}^n \mid Ex \leq x\} , \tag{8}$$

where $E = (-D)^T$. Then we have the following lemma.

Lemma 6 *If the matrix E is irreducible, then the semimodule $\mathcal{K}$ defined by (8) has finite volume. Moreover, if E has spectral radius strictly greater than the unit (that is, 0), then $\mathcal{K}$ reduces to the null vector.*

Proof. In the first place, let us see that $\mathcal{K} = \operatorname{Im}(E^*) \cap \mathbb{Z}_{\max}^n$, where

$$E^* = \bigoplus_{r=0}^{\infty} E^r = I \oplus E \oplus E^2 \oplus \cdots$$

(note that the matrix E^* can have entries equal to $+\infty$, so that E^* should be thought of as a map from $\overline{\mathbb{Z}}_{\max}^n$ to $\overline{\mathbb{Z}}_{\max}^n$). Indeed, we have:

$$\begin{gathered} x \in \mathcal{K} \Rightarrow Ex \leq x, x \in \mathbb{Z}_{\max}^n \Rightarrow \\ E^r x \leq x, \forall r \in \mathbb{N}, x \in \mathbb{Z}_{\max}^n \Rightarrow E^* x \leq x, x \in \mathbb{Z}_{\max}^n \Rightarrow \\ E^* x = x, x \in \mathbb{Z}_{\max}^n \Rightarrow x \in \operatorname{Im}(E^*) \cap \mathbb{Z}_{\max}^n , \end{gathered}$$

and

$$x \in \operatorname{Im}(E^*) \cap \mathbb{Z}^n_{\max} \Rightarrow$$
$$x = E^* y, \text{ for some } y \in \overline{\mathbb{Z}}^n_{\max}, x \in \mathbb{Z}^n_{\max} \Rightarrow$$
$$Ex \leq E^* x = E^* E^* y = E^* y = x, x \in \mathbb{Z}^n_{\max} \Rightarrow x \in \mathcal{K} .$$

When E has spectral radius less than or equal to the unit, we can see that:

$$E^* = I \oplus E \oplus \cdots \oplus E^{n-1} ,$$

since $E^r \leq I \oplus E \oplus \cdots \oplus E^{n-1}$ for all $r \geq n$ (see for example Theorem 3.20 of [BCOQ92]). Moreover, since E is irreducible, we know that all the entries of E^* are finite. Indeed, this is a straightforward consequence of the interpretation of the entries of the matrix E^k in terms of the weight of paths in the precedence graph of E. Then, the proof of Corollary 1 shows that $\mathcal{K}$ has finite volume.

When E has spectral radius strictly greater than the unit, since E is irreducible, all the entries of E^* are equal to $+\infty$ (once again by the interpretation of the entries of the matrix E^k in terms of the weight of paths in the precedence graph of E). Therefore, the only vector in $\mathcal{K} = \operatorname{Im}(E^*) \cap \mathbb{Z}^n_{\max}$ is the null vector. ■

We end this section with an example showing that in Theorem 2, the bound $\operatorname{vol}(\mathcal{K}) + 1$ on the number of steps needed to stabilize the sequence of semimodules $\{\mathcal{X}_r\}_{r\in\mathbb{N}}$ defined by (3), cannot be improved.

Example 3 *Let us consider the matrices*

$$A = \begin{pmatrix} 1 & -\infty \\ -\infty & 0 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 0 \\ 0 \end{pmatrix} ,$$

and the semimodule $\mathcal{K} = \{(x,y)^T \in \mathbb{Z}^2_{\max} \mid x+1 \leq y \leq x+l\}$, where $l \in \mathbb{N}$. Then, in this case we have:

$$\tilde{\mathcal{K}} = \{(x,y)^T \in \mathcal{K} \mid x \oplus y = 0\} = \{(-1,0)^T, \ldots, (-l,0)^T\} ,$$

from which we get $\operatorname{vol}(\mathcal{K}) = l$. Therefore, we are able to apply Theorem 2. In fact, it is straightforward to see that $\mathcal{K} = \operatorname{Im} K$, where

$$K = \begin{pmatrix} 0 & 0 \\ 1 & l \end{pmatrix} ,$$

so we are also in a position to apply Corollary 1.

By Theorem 2 we know that the sequence of semimodules $\{\mathcal{X}_r\}_{r\in\mathbb{N}}$ defined by (3) must stabilize in at most $\operatorname{vol}(\mathcal{K}) + 1 = l + 1$ steps. Let us check this fact in this particular case. In the first place, note that $\mathcal{K} \subset \{(x,y)^T \in \mathbb{Z}^2_{\max} \mid x+1 \leq y\}$, so that $\mathcal{X}_r \subset \mathcal{K} \subset \{(x,y)^T \in \mathbb{Z}^2_{\max} \mid x+1 \leq y\}$ for all $r \in \mathbb{N}$. Then, it is easy to show (applying a straightforward variant *of the computation of $\mathcal{X}_r \ominus \mathcal{B}$ done in Example 2) that $\mathcal{X}_r \ominus \mathcal{B} = \mathcal{X}_r$ for all $r \in \mathbb{N}$. In this way we get:*

$$\begin{aligned}
\mathcal{X}_1 &= \left\{(x,y)^T \in \mathbb{Z}^2_{\max} \mid x+1 \leq y \leq x+l\right\} \ , \\
\mathcal{X}_2 &= \mathcal{X}_1 \cap A^{-1}(\mathcal{X}_1 \ominus \mathcal{B}) = \mathcal{X}_1 \cap A^{-1}(\mathcal{X}_1) = \\
&= \left\{(x,y)^T \in \mathbb{Z}^2_{\max} \mid x+1 \leq y \leq x+l\right\} \cap \\
&\quad\ \left\{(x,y)^T \in \mathbb{Z}^2_{\max} \mid x+2 \leq y \leq x+l+1\right\} = \\
&= \left\{(x,y)^T \in \mathbb{Z}^2_{\max} \mid x+2 \leq y \leq x+l\right\} \subsetneq \mathcal{X}_1 \ , \\
&\ \vdots \\
\mathcal{X}_l &= \mathcal{X}_{l-1} \cap A^{-1}(\mathcal{X}_{l-1} \ominus \mathcal{B}) = \mathcal{X}_{l-1} \cap A^{-1}(\mathcal{X}_{l-1}) = \\
&= \left\{(x,y)^T \in \mathbb{Z}^2_{\max} \mid x+l-1 \leq y \leq x+l\right\} \cap \\
&\quad\ \left\{(x,y)^T \in \mathbb{Z}^2_{\max} \mid x+l \leq y \leq x+l+1\right\} = \\
&= \left\{(x,y)^T \in \mathbb{Z}^2_{\max} \mid x+l \leq y \leq x+l\right\} = \\
&= \left\{(x,y)^T \in \mathbb{Z}^2_{\max} \mid y = x+l\right\} \subsetneq \mathcal{X}_{l-1} \ , \\
\mathcal{X}_{l+1} &= \mathcal{X}_l \cap A^{-1}(\mathcal{X}_l \ominus \mathcal{B}) = \mathcal{X}_l \cap A^{-1}(\mathcal{X}_l) = \\
&= \left\{(x,y)^T \in \mathbb{Z}^2_{\max} \mid y = x+l\right\} \cap \left\{(x,y)^T \in \mathbb{Z}^2_{\max} \mid y = x+l+1\right\} = \\
&= \left\{(-\infty,-\infty)^T\right\} \subsetneq \mathcal{X}_l \ .
\end{aligned}$$

Then, since by Lemma 3 we know that

$$\left\{(-\infty,-\infty)^T\right\} \subset \mathcal{X}_{l+2} \subset \mathcal{X}_{l+1} = \left\{(-\infty,-\infty)^T\right\} \ ,$$

it is clear that $\mathcal{X}_{l+2} = \mathcal{X}_{l+1}$, and therefore

$$\mathcal{K}^* = \mathcal{X}_{l+1} = \left\{(-\infty,-\infty)^T\right\} \ .$$

In this way we see that in this particular case the sequence of semimodules $\{\mathcal{X}_r\}_{r\in\mathbb{N}}$ stabilizes in exactly $\mathrm{vol}\,(\mathcal{K})+1 = l+1$ *steps.*

5 Algebraically (A,B)-invariant semimodules

This section deals with another fundamental problem in the geometric approach to the theory of linear dynamical systems: the computation of a linear feedback. Let us once again consider the dynamical system (1). Let us assume that we already know the maximal (geometrically) (A,B)-invariant semimodule $\mathcal{K}^*$ contained in a given semimodule $\mathcal{K} \subset \mathcal{S}^n$. From a dynamical point of view, this means that the trajectories of system (1) starting in $\mathcal{K}^*$ can be kept inside $\mathcal{K}^*$ by a suitable choice of the control. Our new problem is to determine whether this control can be generated by using a state feedback. In other words, we want to determine whether there exists a linear feedback $u(k) = Fx(k-1)$, where $F \in \mathcal{S}^{q\times n}$, which makes $\mathcal{K}^*$ invariant with respect to the resulting closed loop system:

$$x(k) = (A \oplus BF)x(k-1) \ , \tag{9}$$

that is, such that every trajectory of the closed loop system (9) is completely contained in $\mathcal{K}^*$ when its initial state is in $\mathcal{K}^*$. If a linear feedback with this property exists, we will say that $\mathcal{K}^*$ is an algebraically (A,B)-invariant semimodule. Some authors call this notion $(A+BF)$-invariance (see [Ass99]) or the feedback property (see [Hau82, CP95, CP94]).

Definition 3 *Given the matrices $A \in \mathcal{S}^{n\times n}$ and $B \in \mathcal{S}^{n\times q}$, we say that the semimodule $\mathcal{X} \subset \mathcal{S}^n$ is* algebraically (A,B)-invariant *if there exists $F \in \mathcal{S}^{q\times n}$ such that*

$$(A \oplus BF)\mathcal{X} \subset \mathcal{X} \ .$$

Obviously, every algebraically (A,B)-invariant semimodule is also geometrically (A,B)-invariant. Nevertheless, when $\mathcal{S} = \mathbb{Z}_{\max}$ it is not clear whether a geometrically (A,B)-invariant semimodule is algebraically (A,B)-invariant. Once again, this problem is reminiscent of difficulties of the theory of linear dynamical systems over rings (see [Hau82, Hau84, CP94, CP95, Ass99, ALP99]). Indeed, in the case of linear dynamical systems with coefficients in a field, the class of geometrically (A,B)-invariant spaces coincides with the class of algebraically (A,B)-invariant spaces (see [Won85]). This property makes the (geometrically) (A,B)-invariant spaces very useful in the classical theory. However, this crucial feature is no longer true for linear dynamical systems with coefficients in a ring, that is, there exist geometrically (A,B)-invariant modules which are not algebraically (A,B)-invariant (see [Hau82], in particular Example 2.3). The following example shows that this is also the case for linear dynamical systems over the tropical semiring $\mathbb{N}_{\min} = (\mathbb{N} \cup \{+\infty\}, \min, +)$. In the case of rings, the class of algebraically (A,B)-invariant modules does not have some of the properties of the class of geometrically (A,B)-invariant modules. For example, a module $\mathcal{K}$ does not necessarily have a maximal algebraically (A,B)-invariant submodule (see [Hau82, CP95, CP94]). A necessary and sufficient condition for $\mathcal{K}^*$ to be algebraically (A,B)-invariant can be given in the form of a factorization condition on the transfer function, assuming that the system is reachable and injective (see [Hau82]). When $\mathcal{S}$ is a Principal Ideal Domain, it can be shown that $\mathcal{K}^*$ is algebraically (A,B)-invariant if and only if it is a direct summand (see [Hau82, CP95, CP94]).

Example 4 *Let $\mathcal{S} = \mathbb{N}_{\min}$. Let us consider the matrices*

$$A = \begin{pmatrix} 1 & +\infty \\ 1 & 0 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ ,$$

and the semimodule $\mathcal{K} = \{(x,y)^T \in \mathbb{N}^2_{\min} \mid x \leq y\}$.

In the first place, let us compute the maximal geometrically (A,B)-invariant semimodule $\mathcal{K}^*$ contained in $\mathcal{K}$. With this aim, we will compute the sequence of semimodules $\{\mathcal{X}_r\}_{r\in\mathbb{N}}$

defined by (3). We have:

$$\begin{aligned}
\mathcal{X}_1 &= \mathcal{K} = \left\{(x,y)^T \in \mathbb{N}^2_{\min} \mid x \leq y\right\} = \operatorname{Im} \begin{pmatrix} 0 & 0 \\ +\infty & 0 \end{pmatrix} , \\
\mathcal{X}_2 &= \mathcal{X}_1 \cap A^{-1}(\mathcal{X}_1 \ominus \mathcal{B}) = \\
&= \left\{(x,y)^T \in \mathbb{N}^2_{\min} \mid x \leq y\right\} \cap \left\{(x,y)^T \in \mathbb{N}^2_{\min} \mid 1 \leq y\right\} = \\
&= \left\{(x,y)^T \in \mathbb{N}^2_{\min} \mid x \leq y, 1 \leq y\right\} = \operatorname{Im} \begin{pmatrix} 0 & 1 \\ +\infty & 1 \end{pmatrix} , \\
\mathcal{X}_3 &= \mathcal{X}_2 \cap A^{-1}(\mathcal{X}_2 \ominus \mathcal{B}) = \\
&= \left\{(x,y)^T \in \mathbb{N}^2_{\min} \mid x \leq y, 1 \leq y\right\} \cap \left\{(x,y)^T \in \mathbb{N}^2_{\min} \mid 1 \leq y\right\} = \\
&= \left\{(x,y)^T \in \mathbb{N}^2_{\min} \mid x \leq y, 1 \leq y\right\} = \mathcal{X}_2 .
\end{aligned}$$

Then, we get $\mathcal{K}^* = \mathcal{X}_2 = \left\{(x,y)^T \in \mathbb{N}^2_{\min} \mid x \leq y, 1 \leq y\right\}$. Indeed, it is easy to check that a trajectory which starts at a point of $\mathcal{K} \setminus \left\{(0,0)^T\right\} = \mathcal{K}^*$ can be kept inside $\mathcal{K}$ with the sequence of controls identically equal to $(1,1)^T$, and that a trajectory which starts at the point $(0,0)^T$ cannot be kept inside $\mathcal{K}$ (since for all controls in $\mathcal{B}$ the next state of the system is always $(1,0)^T$, which does not belong to $\mathcal{K}$).

Let us now see that $\mathcal{K}^*$ is not an algebraically (A,B)-invariant semimodule. With this aim, we will show that a trajectory which starts at the point $(1,1)^T \in \mathcal{K}^*$ cannot be kept inside $\mathcal{K}^*$ when a linear state feedback is applied. Let $F \in \mathbb{N}^{1\times 2}_{\min}$ be an arbitrary feedback. Then, since $F(1,1)^T \geq 1$, we know that $BF(1,1)^T = (\alpha,\alpha)^T$, where $\alpha \geq 2$. Therefore,

$$(A \oplus BF)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} \alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \notin \mathcal{K}^* ,$$

which shows that $\mathcal{K}^*$ is not an algebraically (A,B)-invariant semimodule.

We next show how we can decide, using the existing results on max-plus linear equations, whether a finitely generated subsemimodule of $\mathbb{Z}^n_{\max}$ is algebraically (A,B)-invariant. This method also computes a linear feedback with the required property when the subsemimodule is algebraically (A,B)-invariant. Let $A \in \mathbb{Z}^{n\times n}_{\max}$, $B \in \mathbb{Z}^{n\times q}_{\max}$, and let $\mathcal{X}$ be a finitely generated subsemimodule of $\mathbb{Z}^n_{\max}$, so that there exists $Q \in \mathbb{Z}^{n\times r}_{\max}$, for some $r \in \mathbb{N}$, such that $\mathcal{X} = \operatorname{Im} Q$. Then, from Definition 3 it readily follows that $\mathcal{X}$ is an algebraically (A,B)-invariant semimodule if and only if there exist matrices $F \in \mathbb{Z}^{q\times n}_{\max}$ and $G \in \mathbb{Z}^{r\times r}_{\max}$ such that:

$$(A \oplus BF)Q = QG . \tag{10}$$

As (10) is a two sided max-plus linear system of equations, we know that its set of solutions (F,G) is a finitely generated max-plus convex set, which can be explicitly computed by the general elimination methods (see [BH84, Gau92, Gau98, GP97]). In this way we see that we can effectively decide whether a finitely generated subsemimodule of $\mathbb{Z}^n_{\max}$ is algebraically (A,B)-invariant.

Remark 2 *The elimination algorithm shows that the set of solutions of a homogeneous max-plus linear system of the form $Ax = Bx$, where A, B are matrices of suitable dimensions, is a finitely generated semimodule. This algorithm relies on the fact that hyperplanes of $\mathbb{R}^n_{\max}$ (that is, the set of solutions of an equation of the form $ax = bx$, where $a, b \in \mathbb{R}^n_{\max}$ are row vectors) are finitely generated. It is worth mentioning that the resulting naive algorithm has an a priori doubly exponential complexity. However, the doubly exponential bound is pessimistic. It is possible to incorporate in this algorithm the elimination of redundant generators which reduces its execution time. In fact, we are currently working on this subject and we believe that improvements are possible, since we have shown by direct arguments that the number of generators of the set of solutions is at most simply exponential. This will be the subject of a further work.*

Let us note that to decide whether $\mathcal{X} = \operatorname{Im} Q$ is an algebraically (A, B)-invariant semimodule it suffices to know whether the system of equations (10) has at least one solution. Since there exist algorithms to compute only one solution (with finite entries) of a homogeneous max-plus linear system of the form $Ax = Bx$, which seem to be more efficient in practice than the elimination methods (see [BCG03, WB98]), we propose the following alternative procedure. In the first place, we add one unknown t to obtain a homogeneous max-plus linear system of equations:

$$(At \oplus BF)Q = QG \; . \tag{11}$$

Now it is straightforward to see that system (10) has at least one solution if and only if system (11) has at least one solution with $t \neq -\infty$. Therefore, the semimodule $\mathcal{X} = \operatorname{Im} Q$ is algebraically (A, B)-invariant if and only if system (11) has at least one solution with $t \neq -\infty$.

Let us finally recall that the problem of the existence of a solution (with finite entries) of a homogeneous max-plus linear system can be reduced to the problem of the existence of a sub-fixed point of a min-max function (see [Gau98, GG98, CTGG99]). In the case of system (11), using elements of residuation theory (see for example [BCOQ92]), it can be easily shown that (t, F, G) is a solution of (11) if and only if

$$\begin{array}{rcl} t & \leq & (AQ)\backslash(QG) \; , \\ F & \leq & B\backslash(QG)/Q \; , \\ G & \leq & Q\backslash((At \oplus BF)Q) \; , \end{array} \tag{12}$$

where $D\backslash C$ is defined as $\sup\{E \in \overline{\mathbb{Z}}^{p\times r}_{\max} \mid DE \leq C\}$ for all $D \in \mathbb{Z}^{n\times p}_{\max}$ and $C \in \mathbb{Z}^{n\times r}_{\max}$ (the function $/$ is defined in an analogous way). Since $D\backslash C$ can be computed as $(-D^T)C$, where the product is performed in $\overline{\mathbb{Z}}_{\min}$ (see [BCOQ92]), it follows that (12) is a problem of the form $x \leq f(x)$, where f is a min-max function. We see in this way that our problem reduces to a variant of a well-known problem: the existence of a sub-fixed point of a min-max function. For this problem there exist algorithms which behave remarkably well in practice, although their complexities are not yet well understood (see [GG98, CTGG99]). The only difference is that all these algorithms have been developed to find solutions with finite entries, but we are in fact interested in solutions which only have t finite.

6 Application to transportation networks with a timetable

Let us consider the railway network given in Figure 1. Firstly, we will recall how the evolution of this kind of transportation network can be described by max-plus linear dynamical systems of the form of (1). We are interested in the departure times of the trains from the stations. Let $x_i(k)$ denote the k-th departure time of the train which leaves in direction i, for $i = 1, \ldots, n$ (we have $n = 4$ in Figure 1). As we explained in the introduction, a train cannot leave before a number of conditions have been satisfied. A first condition is that the train must have arrived at the station. For instance, let us assume that the train which leaves in direction i is the one which comes from direction $r(i)$. Then the following condition must be satisfied:

$$a_{ir(i)} + x_{r(i)}(k-1) \leq x_i(k) \ , \tag{13}$$

where $a_{ir(i)}$ is the traveling time in direction $r(i)$ (to which the time needed for passengers to leave and board the train is added). A second constraint follows from the demand that trains must connect. This gives rise to the following condition

$$a_{ij} + x_j(k-1) \leq x_i(k) \ , \ \forall j \in C(i) \ , \tag{14}$$

where $C(i)$ is the set of all the directions of the trains which have to provide a connection with the train which leaves in direction i. Finally, the last condition is that a train cannot leave before its scheduled departure time. This yields

$$u_i(k) \leq x_i(k) \ , \tag{15}$$

where $u_i(k)$ denotes the scheduled departure time for the k-th train in direction i. Now, if we assume that a train leaves as soon as all the previous conditions have been satisfied, in max-plus notation conditions (13), (14) and (15) lead to

$$x_i(k) = \bigoplus_{j \in C(i)} a_{ij} x_j(k-1) \oplus a_{ir(i)} x_{r(i)}(k-1) \oplus u_i(k) \ . \tag{16}$$

If we define $a_{ij} = -\infty$ for all $j \notin C(i) \cup \{r(i)\}$, $x(k) = (x_1(k), \ldots, x_n(k))^T$ and $u(k) = (u_1(k), \ldots, u_n(k))^T$, then (16) can be written in matrix form as

$$x(k) = Ax(k-1) \oplus u(k) \ , \tag{17}$$

where $A = (a_{ij}) \in \mathbb{Z}_{\max}^{n \times n}$, which is a system of the form of (1). In the particular case of the railway network shown in Figure 1 we have

$$A = \begin{pmatrix} -\infty & 17 & -\infty & -\infty \\ -\infty & -\infty & 11 & 9 \\ 14 & -\infty & 11 & 9 \\ 14 & -\infty & 11 & -\infty \end{pmatrix} \ .$$

Suppose now that we want to decide whether there exists a timetable such that the time between two consecutive train departures in the same direction is less than a certain given bound or such that the time that people have to wait to make some connections is less than another given bound. To be able to model this kind of requirement it is convenient to introduce the extended state vector $\overline{x}(k) = (x_1(k), \ldots, x_n(k), x_1(k-1), \ldots, x_n(k-1))^T$. Then (17) can be rewritten as $\overline{x}(k) = \overline{A}\overline{x}(k-1) \oplus \overline{B}u(k)$, where

$$\overline{A} = \begin{pmatrix} A & \varepsilon \\ I & \varepsilon \end{pmatrix} \text{ and } \overline{B} = \begin{pmatrix} I \\ \varepsilon \end{pmatrix}$$

(here $I, \varepsilon \in \mathbb{Z}_{\max}^{n \times n}$ denote the max-plus identity and zero matrices, respectively). Assume that we want the time between two consecutive train departures in direction i to be less than L_i time units. This can be expressed as $\overline{x}_i(k) - \overline{x}_{i+n}(k) \leq L_i$, or equivalently as $\overline{x}_i(k) - L_i \leq \overline{x}_{i+n}(k)$. For simplicity we will take the same bound L for all the directions, although everything that follows can done with different bounds. Then the previous condition can be written in matrix form as

$$\begin{pmatrix} \varepsilon & \varepsilon \\ (-L)I & \varepsilon \end{pmatrix} \overline{x}(k) \leq \overline{x}(k) \ , \ \forall k \in \mathbb{N} \ . \tag{18}$$

Suppose now that we want passengers coming from direction i not to have to wait more than M_{ij} time units for the departure of the train which leaves in direction j. This can be expressed as $\overline{x}_j(k) - a_{ji} - \overline{x}_{i+n}(k) \leq M_{ij}$, which is equivalent to $\overline{x}_j(k) - a_{ji} - M_{ij} \leq \overline{x}_{i+n}(k)$. Once again, if for simplicity we take the same bound M for all the possible connections, the previous condition can be written in matrix form as

$$\begin{pmatrix} \varepsilon & \varepsilon \\ (-M)S & \varepsilon \end{pmatrix} \overline{x}(k) \leq \overline{x}(k) \ , \ \forall k \in \mathbb{N} \ , \tag{19}$$

where the matrix $S = (s_{ij}) \in \mathbb{Z}_{\max}^{n \times n}$ is defined by $s_{ij} = -a_{ji}$ if $a_{ji} \neq -\infty$ and $s_{ij} = -\infty$ if $a_{ji} = -\infty$. Finally, the obvious physical constraints $x(k-1) \leq x(k)$ and $Ax(k-1) \leq x(k)$ lead to the following condition for the extended state vector

$$\begin{pmatrix} \varepsilon & I \oplus A \\ \varepsilon & \varepsilon \end{pmatrix} \overline{x}(k) \leq \overline{x}(k) \ , \ \forall k \in \mathbb{N} \ . \tag{20}$$

Therefore, to get the desired behavior of the network, the timetable $u(k)$ should be such that the extended state vector satisfies conditions (18), (19) and (20), that is, such that $E\overline{x}(k) \leq \overline{x}(k)$ for all $k \in \mathbb{N}$, where

$$E = \begin{pmatrix} \varepsilon & I \oplus A \\ (-M)S \oplus (-L)I & \varepsilon \end{pmatrix} \ .$$

For instance, let us take $L = 15$ and $M = 4$ in the case of the railway network shown in Figure 1. Then $E\overline{x}(k) \leq \overline{x}(k)$ is equivalent to $\overline{x}(k) \in \operatorname{Im} E^*$ (see the proof of Lemma 6), where

$$E^* = \begin{pmatrix} 0 & 2 & -2 & -2 & 12 & 17 & 13 & 11 \\ -5 & 0 & -4 & -4 & 10 & 12 & 11 & 9 \\ -1 & 1 & 0 & -3 & 14 & 16 & 12 & 10 \\ -1 & 1 & -3 & 0 & 14 & 16 & 12 & 10 \\ -15 & -13 & -17 & -17 & 0 & 2 & -2 & -4 \\ -20 & -15 & -19 & -19 & -5 & 0 & -4 & -6 \\ -16 & -14 & -15 & -15 & -1 & 1 & 0 & -5 \\ -14 & -12 & -13 & -15 & 1 & 3 & -1 & 0 \end{pmatrix} .$$

Therefore, our problem is to determine the maximal geometrically $(\overline{A}, \overline{B})$-invariant semimodule contained in $\mathcal{K} = \operatorname{Im} E^*$. With this aim we will compute the sequence of semimodules $\{\mathcal{X}_r\}_{r \in \mathbb{N}}$ defined by (3). Since the entries of E^* are all finite, from Corollary 1 we know that this sequence must stabilize. In fact, we can show that $\mathcal{X}_5 = \mathcal{X}_4 \subsetneq \mathcal{X}_3 \subsetneq \mathcal{X}_2 \subsetneq \mathcal{X}_1 = \mathcal{K}$. Then, the maximal geometrically $(\overline{A}, \overline{B})$-invariant semimodule $\mathcal{K}^*$ contained in $\mathcal{K}$ is $\mathcal{X}_4$, which is generated by the columns of the following matrix

$$\begin{pmatrix} 17 & 17 & 17 & 18 & 17 \\ 15 & 15 & 14 & 15 & 15 \\ 18 & 18 & 17 & 18 & 18 \\ 19 & 19 & 18 & 19 & 19 \\ 4 & 2 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & 3 & 4 & 4 \\ 5 & 5 & 4 & 5 & 2 \end{pmatrix} .$$

Consequently, it is possible to obtain the desired behavior of the network with a suitable choice of the timetable $u(k)$ when the initial state belongs to $\mathcal{K}^*$. To be able to compute these timetables we use the method described at the end of Section 5 to decide whether $\mathcal{K}^* = \mathcal{K}_4$ is an algebraically $(\overline{A}, \overline{B})$-invariant semimodule (that is, we apply sub-fixed point techniques to find a state feedback). In this way we can see that $\mathcal{K}^*$ is algebraically $(\overline{A}, \overline{B})$-invariant and one possible state feedback is given by

$$\overline{F} = \begin{pmatrix} 14 & 14 & 14 & 13 & 14 & 14 & 14 & 14 \\ 11 & 14 & 11 & 10 & 14 & 14 & 14 & 14 \\ 14 & 14 & 14 & 13 & 14 & 14 & 14 & 14 \\ 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 \end{pmatrix} .$$

For instance, let us consider the evolution of the network when the initial state is $\overline{x}(0) = (5, 4, 0, 4, 19, 18, 15, 17)^T \in \mathcal{K}^*$ and the control $\overline{F}$ is applied. In that case we obtain the following trajectory $x(k)$ of the system

$$\begin{pmatrix} 4 \\ 0 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 17 \\ 15 \\ 18 \\ 19 \end{pmatrix}, \begin{pmatrix} 32 \\ 29 \\ 32 \\ 33 \end{pmatrix}, \begin{pmatrix} 46 \\ 43 \\ 46 \\ 47 \end{pmatrix}, \begin{pmatrix} 60 \\ 57 \\ 60 \\ 61 \end{pmatrix}, \begin{pmatrix} 74 \\ 71 \\ 74 \\ 75 \end{pmatrix}, \ldots$$

which clearly satisfies the constraints imposed on the network. However, if no control is applied, we get the following trajectory starting from the same initial state

$$\begin{pmatrix}4\\0\\4\\5\end{pmatrix}, \begin{pmatrix}17\\15\\18\\19\end{pmatrix}, \begin{pmatrix}32\\29\\31\\31\end{pmatrix}, \begin{pmatrix}46\\42\\46\\46\end{pmatrix}, \begin{pmatrix}59\\57\\60\\60\end{pmatrix}, \begin{pmatrix}74\\71\\73\\73\end{pmatrix}, \cdots$$

which does not satisfy the constraints imposed on the network, since for example the passengers coming from station S on the third train (which leaves from station Q in direction 4 at time 31) will have to wait 6 time units for the next departure of a train in direction 3 toward station R (which will take place at time 46).

If we want to obtain the desired behavior of the network with a periodic timetable, that is with a timetable $u(k)$ of the form $u(k) = \lambda^k u$, where $\lambda \in \mathbb{Z}_{\max}$ and $u \in \mathbb{Z}^n_{\max}$, then what we can do is to see if the matrix $\overline{A} \oplus \overline{BF}$ has an eigenvector in $\mathcal{K}^*$. In this case it can be shown that $\overline{x}(0) = (4,3,0,3,18,17,14,17)^T \in \mathcal{K}^*$ is an eigenvector of $\overline{A} \oplus \overline{BF}$ corresponding to the eigenvalue $\lambda = 14$, that is, the following equality is satisfied:

$$(\overline{A} \oplus \overline{BF})\overline{x}(0) = 14\overline{x}(0) \; .$$

Therefore, the periodic timetable

$$u(k) = \overline{F}\overline{x}(k-1) = 14^{(k-1)}\overline{F}\overline{x}(0) = 14^{(k-1)} \begin{pmatrix}31\\28\\31\\32\end{pmatrix} = 14^{(k+1)} \begin{pmatrix}3\\0\\3\\4\end{pmatrix}$$

leads to the desired behavior of the network when the initial state is $\overline{x}(0)$. In other words, one train should leave in each direction every 14 time units but the k-th departure time of the trains in direction 1 and 3, respectively in direction 4, should be scheduled 3 time units, respectively 4 time units, after the k-th scheduled departure time of the train in direction 2.

Let us finally mention that the computations of the examples presented in this paper have been checked using the max-plus toolbox of scilab (see [Plu98]).

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